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Reliability bound based on the maximum entropy principle with respect to the first truncated moment †

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Abstract

Various reliability methods have been suggested in the literature, but the bound of an estimated reliability has received less attention. The maximum entropy principle is used to obtain the reliability bound with respect to the first moment truncated for the first time. Compared to the previous methods of probability bounding based on given moments, our method is demonstrated to generate a tight upper bound that is practically useful for engineering applications. Numerical examples have shown that a good upper bound of probability of failure is well obtained up to four given moments, but with more moments a divergence problem can occur.

Keywords: First truncated moment; Probability of failure; Reliability bound; Maximum entropy; Moment based distribution bounding; Upper bound of probability

1. Introduction

The main task of a reliability analysis is to translate the effect of uncertainties on the component or system performance to a corresponding probability as a measure how much some given design requirements are satisfied in the probabilistic point of view. Several reliability methods are available to calculate the probability of failure effectively such as FORM [1] (first order reliability method), SORM [2] (second order reliability method), MCS [3] (Monte Carlo simulation), and moment methods [4-10]. The bound of a probability of failure has less attention in the literature even though the accuracy or efficiency of the reliability methods has been the main issue. Among those few in the literature, two important cases of FORM and MCS are reviewed and the results of sample examples are summarized in the Appendix. The inequality [11] to bound the probability of failure of FORM has been introduced. However, the curvature information of the limit state function has to be available and the bound is often not narrow enough to have practical value. The illustrative example shows this point. The bounds of failure probability of MCS are usually indicated as the confidence intervals in terms of the number of sample size. The sample results are compared with the FORM in the Appendix.

To explore the possibility of getting a sharper bound, our interest is focused on the probability distribution bounding technique with limited moment information. Under an arbitrary number of moment constraints, any probability distributions can be mathematically bounded by employing the theorem of Akhiezer [12]. Based on Akhiezer's theorem, Racz et al. [13] have suggested a moment-based distribution bounding method. Once a reference probability distribution is constructed, it provides the probability bounds at any points. Similarly, the reliability bounds are also easily calculated without excessive computational cost. However, the bounds of the existing methods are mostly too loose to be useful for engineering applications. Since it covers all kinds of PDFs including the discrete PDFs, the loose bound is the natural trade-off. Discrete PDFs are rare in structural applications. Tighter bounds would be obtained by limiting our attention to only continuous PDFs.

MEP [14, 15] (maximum entropy principle) is one of the moment methods and a versatile tool allowing an arbitrary number of moments. MEP generates a natural PDF among the infinite PDF candidates that satisfy the same moment conditions. The higher the order of moment is incorporated, the more convergent the PDF to the true is obtained. However, the finite number of moments gives only partial moment information and therefore PDF estimation in MEP always involves some loss of information which is related to the truncated moments. Given the first four moments, it neglects the fifth and higher order moments. Knowing the upper bound of an estimated probability of failure with respect to the possible

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variations of the truncated moment is important for applications.

MEP is suitable in that it basically produces the continuous PDFs. Based on the MEP, we deal with the reliability bound problem in the optimization perspective. If the ranges of the truncated moments are readily determined, it is possible to find the maximum or minimum reliability with respect to the missing moments. On the one hand, there are too many truncated moments (infinite) for an optimization formulation. One obvious choice is to study the influence of the first moment truncation to the probability of failure. Using the Hankel determinant, the admissible range of the first truncated moment can be determined. Thus, an optimization formulation is derived to set the bound of reliability considering the given moments information as well as the first moment truncated. It is found that the proposed formulation based on the MEP successfully reduces the reliability bound whenever converged. The proposed method was applied to the numerical examples.

2. Probability bounds based on given moments

2.1 Mathematical theory of moment problem

A moment problem is to find a PDF $f(x)$ subject to

$$\mu_k = \int_a^b x^k f(x)dx,\ k = 0,1,\cdots \tag{1}$$

where the k-th moment μ_k and the support a and b of $f(x)$ are given.

The uniqueness and existence of the solution to the moment problem is determined by the Hankel matrices [16]. For the Hamburger problem in which the supports are unbounded $(-\infty,\infty)$, the Hankel determinant is defined as:

$$H_n = \begin{vmatrix} \mu_0 & \cdots & \mu_n \\ \vdots & & \\ \mu_n & \cdots & \mu_{2n} \end{vmatrix}. \tag{2}$$

If $H_n > 0$ in Eq. (2), infinite solutions exist. If $H_n = 0$, the Hamburger problem has a unique solution with respect to the given moments $(\mu_0, \mu_1, ..., \mu_{2n})$. The Hamburger problem always requires $(2n+1)$ number of moments including the 0-th moment μ_0. Otherwise, the solution does not exist.

The moment problem on a finite interval [0,1] is referred to as the Hausdorff problem. Since any finite support $[a,b]$ can be reduced to the normalized support $[0,1]$, the Hausdorff problem is appropriate for most engineering applications.

When only a finite number of moments is available, it becomes a truncated(or reduced) Hausdorff moment problem as follows:

Find a PDF $f(x)$ with the given moment constraints,

$$\mu_k = \int_0^1 x^k f(x)dx,\ k = 0,1,\cdots,N. \tag{3}$$

Table 1. Classification of moment problem.

Support $[a,b]$	Name
$(-\infty,\infty)$	Hamburger problem
$[0,\infty)$	Stieltjes problem
$[0,1]$	Hausdorff problem

Given a finite set of moments $(\mu_0, \mu_1, ..., \mu_N)$, a necessary and sufficient condition for the existence of a solution to the truncated Hausdorff problem is

$$\underline{H}_n > 0 \text{ and } \overline{H}_n > 0 \quad \text{for all } n = 1,2,...,N \tag{4}$$

where the Hankel determinants $\underline{H}_n$ and $\overline{H}_n$ are defined as follows:

$$\text{i) } n = 2m,\ \underline{H}_n = \begin{vmatrix} \mu_0 & \cdots & \mu_m \\ \vdots & & \\ \mu_m & \cdots & \mu_{2m} \end{vmatrix}, \quad \overline{H}_n = \begin{vmatrix} \mu_1-\mu_2 & \cdots & \mu_m-\mu_{m+1} \\ \vdots & & \\ \mu_m-\mu_{m+1} & \cdots & \mu_{2m-1}-\mu_{2m} \end{vmatrix}. \tag{5}$$

$$\text{ii) } n = 2m+1,\ \underline{H}_n = \begin{vmatrix} \mu_1 & \cdots & \mu_{m+1} \\ \vdots & & \\ \mu_{m+1} & \cdots & \mu_{2m+1} \end{vmatrix}, \quad \overline{H}_n = \begin{vmatrix} \mu_0-\mu_1 & \cdots & \mu_m-\mu_{m+1} \\ \vdots & & \\ \mu_m-\mu_{m+1} & \cdots & \mu_{2m}-\mu_{2m+1} \end{vmatrix}. \tag{6}$$

Equating $\overline{H}_{N+1}$ =0 and $\underline{H}_{N+1}$ =0 gives the following relationships [16]:

$$\mu_{N+1}^{+} = \mu_{N+1} + \frac{\overline{H}_{N+1}}{\overline{H}_{N-1}}, \quad \mu_{N+1}^{-} = \mu_{N+1} - \frac{\underline{H}_{N+1}}{\underline{H}_{N-1}}. \tag{7}$$

It is noted that μ_{N+1} is cancelled out in the right side of Eq. (7) such that μ_{N+1}^{+} and μ_{N+1}^{-} can be expressed by the given moments $\boldsymbol{\mu}_N = (\mu_0, \mu_1, ..., \mu_N)$. Eq. (7) is used for determining the admissible range(upper and lower bound) of the next moment.

2.2 Moment based probability bounding technique

For any discrete and continuous random variables, the following inequality due to Chebyshev holds [17]. For a random variable X with mean μ_X and standard deviation σ_X, it is given by,

$$\Pr(|X - \mu_X| \geq k\sigma_X) \leq \frac{1}{k^2} \tag{8}$$

where $k > 0$ is a real number. Since Chebyshev's inequality is usually not very sharp, the use of this interval is limited.

A moment-based inequality has been generalized by Akhiezer [12]. Recently, Racz et al. [13] suggested a moment-based distribution bounding method based on Akhiezer's theorem. Let Ω_N be a set of probability distributions. It consists of all distributions which satisfy the same moment constraints:

$$\mu_k = \int_{-\infty}^{\infty} x^k dF(x), k = 0,1,...,N \tag{9}$$

where $F(x)$ is a CDF(cumulative distribution function). The upper and lower bound may be expressed as:

$$p^L \equiv \min_{F(\cdot)\in\Omega_N} F(x), p^U \equiv \max_{F(\cdot)\in\Omega_N} F(x) \tag{10}$$

Using Akhiezer's theorem, we can construct the bounds of probability of failure based on the reference distribution $F^*(x)$ as follows [12, 13]:

$$\int_{-\infty}^{0^-} dF^*(x) \le \int_{-\infty}^{0-} dF(x) \le \int_{-\infty}^{0+} dF(x) \le \int_{-\infty}^{0^-} dF^*(x) + p \tag{11}$$

where

$$p = \frac{\begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}}{\begin{vmatrix} \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \mu_3 & \mu_4 & \cdots & \mu_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n+1} & \mu_{n+2} & \cdots & \mu_{2n} \end{vmatrix}} \tag{12}$$

and $F^*(x)$ has the probability p at 0 and $F^*(x) \in \Omega_N$.

The reference distribution $F^*(x)$ is obtained when $H_n = 0$. This is the case when the distribution consists of n points. The corresponding points $x_i (i = 1,...,n)$ are determined by the roots of the polynomial $P_n(x)$ as:

$$P_n(x) = \frac{1}{\sqrt{H_{n-1}H_n}} \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix}, \tag{13}$$

where $P_0(x) = 1$ and H_n is the Hankel determinant as defined in Eq. (2).

The reference distribution associated with x_i is defined as:

$$1 = p + \sum_{i=1}^{n} p_i, \quad \mu_k = \sum_{i=1}^{n} x_i^k p_i, \quad (k = 1,2,...,n-1) . \tag{14}$$

It can be expressed in a matrix form as:

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} 1-p \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{n-1} \end{pmatrix} . \tag{15}$$

When $\mu_0 = 1, \mu_1$ and μ_2 are prescribed and $H_1 > 0$, the bounds of probability at 0 are given by Racz et al. [13] as follows:

$$\begin{aligned} &p^L = \frac{\mu_2 - \mu_1^2}{\mu_2}, p^U = 1, && \text{if } \mu_1 < 0, \\ &p^L = 0, \quad p^U = 1, && \text{if } \mu_1 = 0, \\ &p^L = 0, \quad p^U = \frac{\mu_2 - \mu_1^2}{\mu_2}, && \text{if } \mu_1 > 0. \end{aligned} \tag{16}$$

When $\mu_0 = 1, \mu_1, \mu_2, \mu_3$ and μ_4 are prescribed and $H_1 > 0$ and $H_2 > 0$, the symbolic bounds are taken [13].

$$p = \frac{-\mu_2^3 + 2\mu_1\mu_2\mu_3 - \mu_3^2 - \mu_1^2\mu_4 + \mu_2\mu_4}{\mu_2\mu_4 - \mu_3^2},$$

$$x_1 = \frac{\mu_2\mu_3 - \mu_1\mu_4 - q}{2\left(\mu_2^2 - \mu_1\mu_3\right)},$$

$$p_1 = \frac{\left(\begin{array}{c} -\mu_2^4\mu_3 + 2\mu_1^2\mu_3^3 + 3\mu_1\mu_2^3\mu_4 - 5\mu_1^2\mu_2\mu_3\mu_4 \\ +\mu_1^3\mu_4^2 - q(\mu_2^3 - 2\mu_1\mu_2\mu_3 + \mu_1^2\mu_4) \end{array}\right)}{2q(\mu_3^2 - \mu_2\mu_4)},$$

$$x_2 = \frac{\mu_2\mu_3 - \mu_1\mu_4 + q}{2\left(\mu_2^2 - \mu_1\mu_3\right)},$$

$$p_2 = -\frac{\mu_2^2 - \mu_1\mu_3}{q} \times \left(-\mu_1 - \frac{\left(\mu_2^3 - 2\mu_1\mu_2\mu_3 + \mu_1^2\mu_4\right)\left(-\mu_2\mu_3 + \mu_1\mu_4 + q\right)}{2(\mu_2^2 - \mu_1\mu_3)\left(-\mu_3^2 + \mu_2\mu_4\right)} \right),$$

where

$$q = \sqrt{(-\mu_2\mu_3 + \mu_1\mu_4)^2 - 4\left(\mu_2^2 - \mu_1\mu_3\right)\left(\mu_3^2 - \mu_2\mu_4\right)}. \tag{17}$$

$$\begin{aligned} &p^L = p_1 + p_2, \quad p^U = 1, && \text{if } x_1 < 0, x_2 < 0, \\ &p^L = p_1, \quad p^U = p_1 + p && \text{if } x_1 < 0, x_2 > 0, \\ &p^L = 0, \quad p^U = p, && \text{if } x_1 > 0, x_2 > 0. \end{aligned} \tag{18}$$

There are no restrictions or assumptions(e.g. unimodal or continuous) on the PDFs in this method. It means that an extreme PDF, for example a discrete PDF, can be used for constructing the reference distribution $F^*(x)$. Consequently, it provides very conservative(loose) bounds although it is mathematically rigorous. It is noted that the method based on

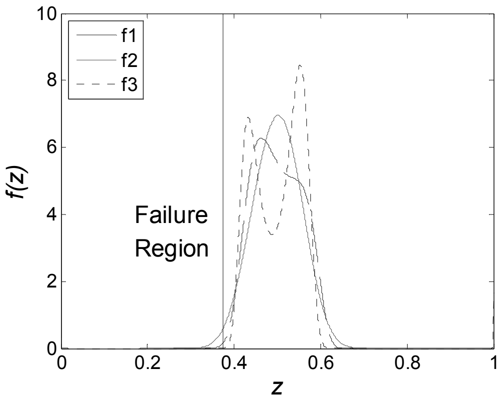

Fig. 1. Three different PDFs by MEP with the same four moments.

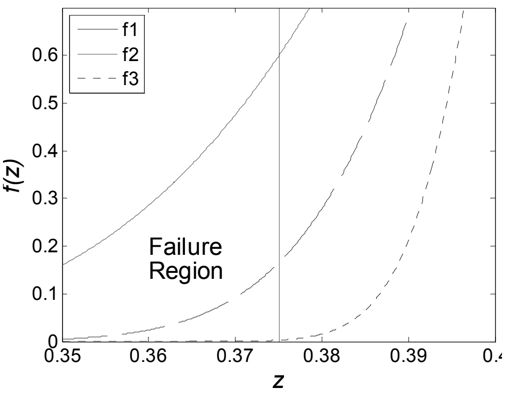

Fig. 2. Tail shapes of three PDFs.

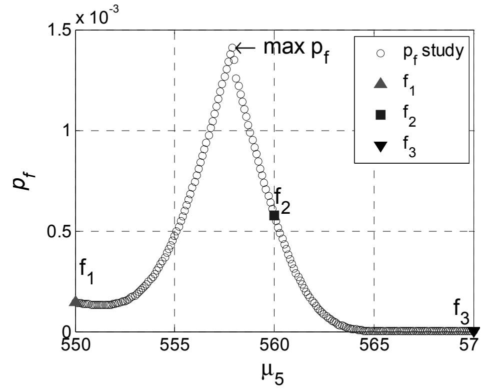

Fig. 3. Parametric study of probability of failure with respect to the variation of μ_5 = [550,570].

Akhiezer's theorem requires an even number of moments(Hamburger problem).

3. Proposed method for reliability bound

3.1 Effect of higher moments truncation

We assume that the first four moments are exactly given and higher moments are not available. Fig. 1 depicts that MEP recovers three PDFs f_1, f_2, f_3 from the different fifth moments values. As shown in Fig. 1 and Table 2, f_1, f_2, f_3 have

Table 2. Moment and PDF comparison between three different PDFs.

	f_1	f_2	f_3
Given: 1st to 4th moment	$\mu_0 = 1$, $(\mu_1, \mu_2, \mu_3, \mu_4) = (3,10,36,138)$		
Probability of failure from the first four moments $(p_f)_4 = \int f_4(x)dx$	1.3499×10^{-3}		
5th moment	$\mu_5 = 550$	$\mu_5 = 560$	$\mu_5 = 570$
Probability of failure from the first five moments $(p_f)_5 = \int f_5(x)dx$	1.4386×10^{-4}	5.7779×10^{-4}	2.3223×10^{-18}

different shapes as well as probabilities. Table 2 implies that the probability of failure can significantly change with respect to the truncated moment μ_5 . Adding an arbitrarily chosen fifth moment into MEP may not provide a confident result. It is safer to examine the worst case in terms of reliability, that is, the maximum probability of failure pertaining to the possible variations of the first truncated moment μ_5 .

A parametric study with respect to μ_5 shows probability variations in Fig. 3. The problem of finding an upper bound can thus be formulated as an optimization to find the maximum probability of failure(minimum reliability) in connection with μ_5 . MEP is fairly useful in that the sensitivity of the probability with respect to the moment is easily calculated. The next section will explain the MEP and the proposed formulation.

3.2 Maximum entropy principle (MEP)

Out of all PDFs satisfying a set of moment constraints, MEP is to choose a PDF that maximizes the entropy measure as follows:

$$\text{maximize}: -\int_a^b f(x)\ln f(x)dx \tag{19}$$

$$\text{subject to}: \int_a^b f(x)dx = 1 \tag{20}$$

$$\int_a^b w_i(x)f(x)dx = r_i, \quad i = 1,2,...,N. \tag{21}$$

Using the Lagrange multiplier, we have the Lagrangian,

$$L = -\int_a^b f(x)\ln f(x)dx - (\lambda_0 - 1)\left[\int_a^b f(x)dx - 1\right] - \sum_{i=1}^{N}\lambda_i\left[\int_a^b w_i(x)f(x)dx - r_i\right]. \tag{22}$$

Use of the Euler-Lagrange equation of the calculus of variations gives [18]

$$f_N(x) = \exp\left(-\lambda_0 - \sum_{i=1}^{N} \lambda_i w_i(x)\right) \tag{23}$$

where $f_N(\cdot)$ denotes the maximum entropy PDF from N moments. When $w_i(x) = x^i$ and $r_i = \mu_i$, the solution is

$$f_N(x) = \exp\left(\sum_{i=0}^{N} -\lambda_i x^i\right). \tag{24}$$

Since the Lagrange multipliers cannot be obtained analytically except for simple cases, a potential function $\Gamma(\boldsymbol{\lambda})$ has been introduced to determine the Lagrange multipliers [19],

$$\Gamma(\boldsymbol{\lambda}) = \lambda_0 + \sum_{i=1}^{N} \lambda_i \mu_i \tag{25}$$

It can be solved by an iterative Newton's algorithm.

$$\lambda_i^{(k+1)} = \lambda_i^{(k)} - \left(\Gamma^{(2)}\right)^{-1} \Gamma^{(1)} \tag{26}$$

where

$$\Gamma^{(1)} = \frac{\partial \Gamma}{\partial \lambda_i} = \mu_i - \int x^i \exp(-\sum_{i=0}^{N} \lambda_i x^i) dx = \mu_i - \tilde{\mu}_i(\boldsymbol{\lambda}) \tag{27}$$

and

$$\begin{aligned} \Gamma^{(2)} &= \frac{\partial^2 \Gamma}{\partial \lambda_i \partial \lambda_j} = \tilde{\mu}_{i+j}(\boldsymbol{\lambda}), \\ \tilde{\mu}_{i+j}(\boldsymbol{\lambda}) &= \int x^{i+j} \exp(-\sum_{i=0}^{N} \lambda_i x^i) dx, \; i,j = 0,1,2,\cdots,N \end{aligned} \tag{28}$$

Eq. (26) can be written in a matrix form,

$$\begin{Bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_N \end{Bmatrix}^{k+1} = \begin{Bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_N \end{Bmatrix}^{k} - \begin{bmatrix} \tilde{\mu}_0 & \tilde{\mu}_1 & \cdots & \tilde{\mu}_N \\ \tilde{\mu}_1 & \tilde{\mu}_2 & \cdots & \tilde{\mu}_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mu}_N & \tilde{\mu}_{N+1} & \cdots & \tilde{\mu}_{2N} \end{bmatrix}^{-1} \begin{Bmatrix} \mu_0 - \tilde{\mu}_0 \\ \mu_1 - \tilde{\mu}_1 \\ \vdots \\ \mu_N - \tilde{\mu}_N \end{Bmatrix}. \tag{29}$$

It is noted that the given moments are normalized for a unit support $[0,1]$. The random variable X (support: $[a,b]$) is transformed to the normalized random variable Z by:

$$Z = \frac{X-a}{b-a} \tag{30}$$

The relation between the moment of X and Z is given as follows:

$$\mu_{z,i} = \frac{1}{(b-a)^i} \sum_{j=0}^{i} \binom{i}{j} (-a)^j \mu_{x,i-j}, \tag{31}$$

$$\mu_{x,i} = \sum_{j=0}^{i} \binom{i}{j} a^{i-j} (b-a)^j \mu_{z,j}, \tag{32}$$

where $\mu_{x,i}$ and $\mu_{z,i}$ are the i -th order moment of support $[a,b]$ and $[0,1]$, respectively.

The derivative of the probability of failure with respect to the i -th order moment can be written using the chain rule,

$$\frac{\partial p_f}{\partial \mu_{x,i}} = \sum_{k=0}^{N} \frac{\partial p_f}{\partial \lambda_k} \sum_{j=0}^{N} \frac{\partial \lambda_k}{\partial \mu_{z,j}} \sum_{i=0}^{N} \frac{\partial \mu_{z,j}}{\partial \mu_{x,i}} \tag{33}$$

The first derivative term $\frac{\partial p_f}{\partial \lambda_k}$ is given as

$$\frac{\partial p_f}{\partial \lambda_k} = -\int_0^{\alpha} z^k \exp(-\sum_{k=0}^{N} \lambda_k z^k) dz \tag{34}$$

where $\alpha = -\frac{a}{b-a}$.

The derivative of the k -th Lagrange multiplier with respect to the j -th normalized moment is

$$\frac{\partial \lambda_k}{\partial \mu_{z,j}} = -C_{jk}$$
$$\text{where } \mathbf{C} = \begin{bmatrix} \tilde{\mu}_0 & \tilde{\mu}_1 & \cdots & \tilde{\mu}_N \\ \tilde{\mu}_1 & \tilde{\mu}_2 & \cdots & \tilde{\mu}_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mu}_N & \tilde{\mu}_{N+1} & \cdots & \tilde{\mu}_{2N} \end{bmatrix}^{-1} \tag{35}$$

3.3 Proposed optimization formulation

The objective is to find the worst case(maximum probability of failure) concerning μ_{N+1}. Therefore, μ_{N+1} is taken as a design variable. The Hankel determinant is used to define the admissible range of μ_{N+1} (Eq. (7)). Our proposed optimization formulation is stated as follows:

$$\text{max.} \left(p_f\right)^{*}_{\text{MEP}-N} \text{ with respect to } \mu_{N+1} \tag{36}$$

$$\text{subject to } \mu_{N+1}^{-} \le \mu_{N+1} \le \mu_{N+1}^{+} \tag{37}$$

where $\left(p_f\right)^{*}_{\text{MEP}-N} = \int_{-\infty}^{0} f_{N+1}(x;\boldsymbol{\mu}_{N+1}) dx$.

The moment vector $\boldsymbol{\mu}_{N+1}$ consists of the given moments $(\mu_0, \mu_1, \cdots, \mu_N)$ and an arbitrary value of μ_{N+1} within $[\mu_{N+1}^{-}, \mu_{N+1}^{+}]$. From Eqs. (33)-(35), the sensitivity information of $(dp_f / d\mu_{N+1})$ is available as soon as the MEP solution is obtained and used for optimization.

3.4 Calculation of statistical moments

A moment-based quadrature rule(MBQR) [20] is available to calculate the integration nodes and weights of the Gauss quadrature rule. It can be, however, numerically unstable when the number of integration nodes increases [9]. In this paper, the nodes and weights of the Gauss quadrature rule are computed from the recursion coefficients relating the orthogonal polynomial with respect to a specified PDF [21-23]. More details and numerical algorithms are referred to [21, 24].

3.5 Determination of admissible range of moment

When the moments are assigned, the condition for existence of an MEP solution is discussed in [25]. It is identical to the condition that the corresponding truncated Hausdorff moment problem admits a solution, i.e., the Hankel determinants must be positive. Hence, using MEP, the boundary for the design variable can cover the whole domain of μ_{N+1} that is determined by Eq. (7). There are some guidelines for limiting the range of the moment. First, the Hankel determinant should be strictly positive. Second, moderate tolerances are needed at the bounds of μ_{N+1}. Theoretically, the solutions of MEP are obtainable at the bounds of μ_{N+1}, but numerical instability can occur at both bounds. Tolerances help us to avoid numerical instability.

The possible region of kurtosis in the Pearson system is different from that of MEP. Its upper and lower bounds are given by [26],

$$\beta_1 + 1 < \beta_2 < \frac{15\beta_1 + 36}{8} \tag{38}$$

where β_1 and β_2 is the skewness and kurtosis of limit state function, respectively. The lower bound of kurtosis of the Pearson system is the same as that of MEP, but its upper bound is much tighter than that of MEP because of the additional assumption in the Pearson system.

3.6 Optimization procedure

Fig. 4 depicts the overall optimization procedure. A preliminary step is needed to obtain the current reliability from the given N moments and then the procedure starts to bound reliability. The algorithm may be summarized thus:

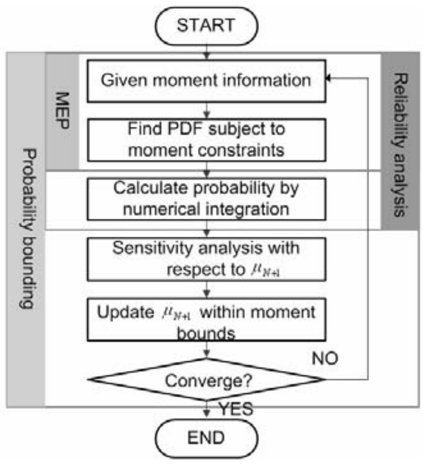

Fig. 4. Flow chart of the optimization for reliability bounds.

Step 1: Given moment information $\mu_1, \cdots, \mu_N$ and an initial value of the moment μ_{N+1} is approximated by $\mu_{N+1} \cong \int_{-\infty}^{\infty} x^{N+1} f_N(x)dx$.

Step 2: Find probability distribution function $f_{N+1}(x)$ based on MEP.

Step 3: Numerical integration is performed to estimate reliability.

Step 4: Calculate the sensitivity of p_f with respect to μ_{N+1} and update the new value.

Step 5: An optimization is considered converged if the difference between the reliability values of two successive iterations is less than a given tolerance. The tolerance for convergence is taken 10^{-6} .

4. Numerical example

Our method is compared with the bound based on the Akhiezer's theorem. First, given N moments $(\mu_1, \mu_2, ... \mu_N)$, the probability of failure based on MEP is estimated. It is denoted as MEP- N . Using the proposed optimization formulation, the upper bound of probability of failure in regard to μ_{N+1} is evaluated based on the MEP. It is denoted as MEP-

Table 3. Random variable parameters (example 1).

	Distribution	Mean	Std.
X_1	Normal	5	0.5
X_2	Normal	5	0.4

Table 4. The upper bound and probability of failure according to the given moments (example 1).

	p_f		p_f^U		p_f^U
MCS[a]	0.055692	MCS ub[b]	0.056142		
MEP-2	0.066271	MEP-2*	0.071489	AK-2	0.3065
MEP-3	0.056326	MEP-3*	0.061725	AK-3	·
MEP-4	0.055146	MEP-4*	0.055286	AK-4	0.2109
MEP-5	0.055245	MEP-5*	Diverged	AK-5	·

a. 10^6 simulation, 95% CI = [0.000594, 0.000694]
b. MCS upper bound(95% confidence)

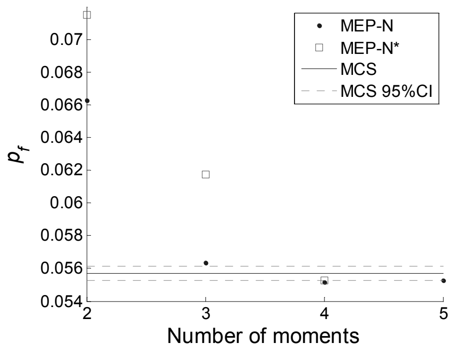

Fig. 5. Upper bound of probability of failure with respect to the number of given moments.

N^*. Following the Akhiezer's theorem [12, 13], another upper bound is constructed depending on the given N moments and denoted as AK-*N*.

4.1 A simple mathematical problem

The limit state function considered and input random variables are described in the following:

$$g(\mathbf{x}) = \frac{80}{x_1^2 + 8x_2 + 5} - 1 \text{ where } p_f = \Pr\left[g(\mathbf{x}) \le 0\right] \quad (39)$$

We can expect that this response is close to a normal distribution, because $(\beta_1, \beta_2) = (0.3364,\ 3.2263)$. The probability of failure MEP- N and the upper bound MEP- N^* are estimated for $N = 2,3,4,5$ in Table 4.

Given more moment information, the MEP result approaches to the MCS probability of failure and the upper bound is also reduced as shown in Fig. 5. Particularly, the upper bound of probability of failure MEP-4*(=0.055286) is almost coincident with the probability of failure MEP-4(=0.055146). The variation is about 1.4×10^{-4} and thus the first four moments are thought to be appropriate for a convergent result. The probability of failure appears to be almost insensitive to μ_5. In the author's experience, it is frequent when the limit state function is not highly nonlinear.

It is shown that the upper bound MEP-4* provides a very tight value(=0.055286), reduced by about 96% compared to the upper bound AK-4(=0.2109). The maximum entropy solution can be numerically obtained with the first five moments, but a divergence problem occurs in obtaining the upper bound of the probability of failure with respect to the sixth moment μ_6.

This is because μ_6 is transformed to a small value due to the support normalization, and the Hessian matrix inversion of MEP in Eq. (29) becomes highly ill-conditioned. Tagliani [27] discussed the stability issue of MEP when the moments $(\mu_0, \cdots \mu_N)$ are fixed and only μ_{N+1} varies. As N is increased, the condition number of Hessian matrix becomes significantly large and the solution of MEP is sensitive to the small variation of μ_{N+1}. It hinders us from obtaining the solution of MEP at some specific values of μ_6.

4.2 Long tail distribution

We consider the long tail problem [28] as follows:

$$g(\mathbf{x}) = 1 - (y-6)^2 - (y-6)^3 + 0.6(y-6)^4 - z \quad (40)$$

where $p_f = \Pr[g(\mathbf{x}) \le 0]$ and $y = 0.9063x_1 + 0.4226x_2$, $z = 0.4226x_1 - 0.9063x_2$.

The statistical properties of the input random variables are given in Table 5. This example is known to have a long tail. Its skewness and kurtosis are given as $(\beta_1, \beta_2) = (2.5036,\ 13.6946)$. Since the kurtosis exceeds the upper bound in Eq. (38), the Pearson system is not applicable.

Table 5. Random variable parameters (example 2).

	Distribution	Mean	Std.
X_1	Normal	4.5580	0.3
X_2	Normal	1.9645	0.3

Table 6. The upper bound and probability of failure according to the given moments (example 2).

	p_f		p_f^U		p_f^U
MCS[a]	0.000644	MCS ub[b]	0.000694		
MEP-2	0.101854	MEP-2*	0.103987	AK-2	0.3823
MEP-3	0.063569	MEP-3*	0.069752	AK-3	·
MEP-4	0.028688	MEP-4*	0.033051	AK-4	0.2058
MEP-5	0.004801	MEP-5*	Diverged	AK-5	·
MEP-6	0.000858				

a. 10^6 simulation, 95% CI = [0.000594, 0.000694]
b. MCS upper bound(95% confidence)

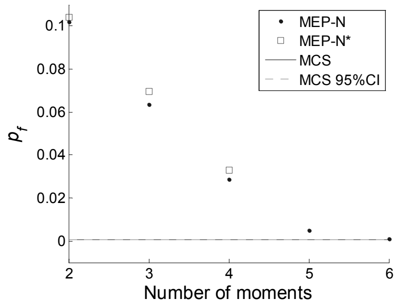

Fig. 6. Upper bound of probability of failure with respect to the number of given moments.

Again, the accuracy of MEP is improved and the bounds also become tighter as the number of given moments is increased as shown in Table 6. The upper bound has been successfully calculated for each case, but it is not as tight as the first example. Moreover, Table 6 shows that MEP-6 (=0.000857) does not have acceptable accuracy in comparison with the MCS result(=0.00064). It is found that the calculated probabilities change drastically as the number of moments increases. When the four moments are accommodated in MEP, the probability of failure MEP-4 is given as 0.028688 and the proposed upper bound is 0.033051. The probability change is about 4.4×10^{-3}. The gap between MEP-4* and MEP-4 is relatively bigger than that of the first example. It indicates that the second example is more sensitive to the truncation of μ_5, that is, μ_5 is not ignorable in reliability estimation. Compared to the upper bound of AK-4(=0.2058), MEP-4* suggests the upper bound as 0.033051, which is almost 84% reduced value. It demonstrates again that our method gives tighter bounds than those by existing bounding in the long tail distribution. It is noticeable that the proposed method can give the upper bound either when N is odd or even, but N must be even in the existing method.

5. Conclusions

In this paper, an optimization formulation is devised to find the upper bound of probability of failure with respect to the first truncated moment. Introducing MEP as a probability bounding tool, our method is shown to provide a tighter bound than the existing techniques. The first four moments are commonly used in the moment methods and they are mostly adequate to obtain the accurate results. Example 1 is taken for considering such a case. Given the first four moments, the difference between the upper bound(MEP-4*) and the current probability of failure(MEP-4) is found to be very small. It means that there remains little possibility to cause a significant change of probability even if the fifth moment is added. In the long tail example, it is shown that the higher order moments are required for good result. The gap between the upper bound and the current probability of failure can be a good indication whether the adoption of the higher order moment is recommended or not. A few observations from the study may be summarized with some discussions as follows:

(1) Given moment $\mu_0,...,\mu_N$, the admissible range of μ_{N+1} should be precisely determined. In this paper, the Hankel determinant is imposed to find the bounds of μ_{N+1}. Since numerical errors are involved in the calculation, some tolerances are applied to moment bounds to avoid potential numerical instability especially around the boundary of the range.

(2) To explore the upper bound depending on the moment truncation, we consider only the exponential function of a linear combination of polynomial basis. By narrowing down the scope, we could have obtained very tight bounds, which were unavailable up to now.

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Appendix: Bounds of probability estimated by FORM and MCS

A.1 FORM and its bounds

FORM [1] is the most widely used method in reliability based design optimization, but its bounds have not been studied extensively. Shinozuka [11] has introduced the following result in the literature referring to an unpublished work of Hasofer. Under the assumption that random variables are Gaussian and the limit state function is a well-behaved concave surface toward the origin, it can be shown that

$$1-\chi_n^2(\beta^2) \geq p_f \geq 1-\Phi(\beta) \tag{A.1}$$

where $\chi_n^2(\bullet)$ and $\Phi(\bullet)$ are chi-square distribution function with n degrees of freedom and standard normal cumulative density function(CDF), respectively.

Fig. A.1 shows the upper and lower bound of probability of failure with respect to β when the number of variables n is 2. The lower and upper bound have an order of difference between them.

When β is fixed, the lower bound $1-\Phi(\beta)$ is also fixed and the upper bound $1-\chi_n^2(\beta^2)$ only depends on the number of random variables n as shown in Fig. A.2.

Eq. (A.1) is simple and easy to apply, but the decision of convexity or concaveness of the limit state function is complicated. Furthermore, the bound is not narrow enough to have practical value. As shown in Fig. A.1 and Fig. A.2, this situation becomes worse when n and β are large. For example, when β =3 and n =20, the lower and the upper bound calculated by Eq. (A.1) is 1.35×10^{-3} and 9.83×10^{-1}, respectively. In this case, the upper bound is almost close to 1, which is not very meaningful.

A.2 MCS and its bounds

MCS [3] is very often used to check the accuracy of reliability analysis as a reference value. Though MCS is generally regarded as the most accurate method, it has also variability owing to the limited sample size. With a sample size of N_{MCS}, the failure probability by MCS is given with the $(1-\alpha)\times100\%$ CI(confidence interval) as follows [3]:

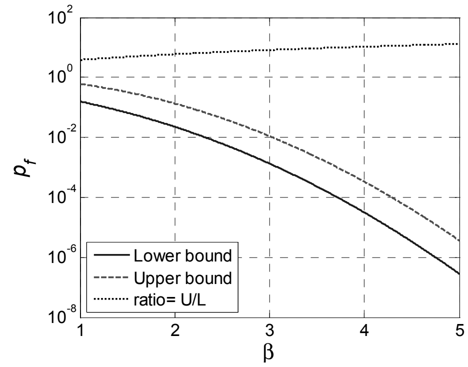

Fig. A.1. Probability of failure bounds of FORM according to reliability index (n =2).

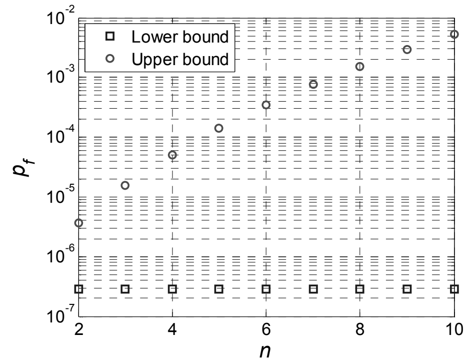

Fig. A.2. Probability of failure bounds of FORM according to the number of random variables (β =3).

$$\Pr\left[\begin{array}{l} p_f^T - z_{1-\alpha/2}\sqrt{\dfrac{(1-p_f^T)p_f^T}{N_{\text{MCS}}}} < \\ \quad p_{f\_\text{MCS}} < p_f^T + z_{1-\alpha/2}\sqrt{\dfrac{(1-p_f^T)p_f^T}{N_{\text{MCS}}}} \end{array}\right] = 1-\alpha \tag{A.2}$$

where p_{f_MCS} and p_f^T denotes the MCS and the true probability of failure, respectively. The $(1-\alpha)\times$100th percentile of the distribution is denoted by z_α such that $\Phi(z_\alpha)=1-\alpha$.

The following example is taken for illustration:

$$g(\mathbf{x}) = 1.5\left(\frac{x_1}{x_2}\right)-1 \tag{A.3}$$

where $p_f = \Pr[g(\mathbf{x})\leq 0]$ and $X_1, X_2 \sim N(5,0.5)$

MCS is conducted for Eq. (A.3) with respect to different sample sizes of $10^3 \sim 10^7$. The 95% CI and p_{f_MCS} vary according to the number of samples N_{MCS} as shown in Fig. (A.3). When N_{MCS} is not sufficiently large, the MCS result is not good and the bounds are not close to each other. For example, when $N_{\text{MCS}}=10^3$, the upper bound 4.77×10^{-3} is about 2.4 times that of p_{f_MCS} $(=2.0\times10^{-3})$. In this case, p_{f_MCS} may not be acceptable as a reference value. From Eq. (A.2), we can infer that the bounds cannot be tight when the MCS sampling number is small. In that case, the sampling

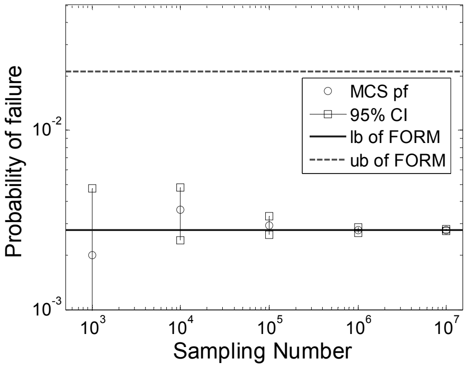

Fig. A.3. MCS probability of failure and 95% CI according to the sampling number.

number should properly be increased until the confidence interval is smaller than a desired tolerance.

The lower and upper bounds of FORM given by Eq. (A.1) are added in Fig. (A.3). When the limit state function is concave toward the origin and the input random variables are Gaussian, the estimated probability of failure by FORM becomes a lower bound. The upper bound, 2.17×10^{-2}, given by Eq.(A.1), is much bigger than that by the 95% confidence interval of MCS.

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